Sampling Theorem Associated with Multiple-parameter Fractional Fourier Transform

Qiwen Ran

National Key Laboratory of Tunable Laser Technology, Harbin Institute of Technology, Harbin, China
Science Research Center, Research Academy of Science and Technology, Harbin Institute of Technology, Harbin, China
Email: qiwenran@hit.edu.cn

Hui Zhao, Guixia Ge, Jing Ma and Liying Tan

National Key Laboratory of Tunable Laser Technology, Harbin Institute of Technology, Harbin, China
Email: {zhaovhit, geguixiahit}@yahoo.com.cn, {majing, tanliying}@hit.edu.cn

Abstract—**We propose a new method for analysis of the sampling and reconstruction conditions of signals by use of the multiple-parameter fractional Fourier transform (MPFRFT). It is shown that the MPFRFT may provide a novel understanding of sampling process. The proposed sampling theorem generalizes classical Shannon sampling theorem and Fourier series expansion, and provides a full-reconstruction procedure of certain signals that are not bandlimited in the conventional Fourier transform domain. An orthogonal basis for the class of signals which are bandlimited in the MPFRFT domain is also given. Experimental results are proposed to verify the accuracy and effectiveness of the obtained results.**

Index Terms—**sampling theorem, fractional Fourier transform, multiple-parameter fractional Fourier transform**

I. INTRODUCTION

In recent years, the concept of fractional operator has been investigated extensively in many engineering applications and science [1-5]. Fractional operators are defined as fractionalizations of some commonly used operators. In this paper, the fractional Fourier transform (FRFT) are considered. The FRFT, as a generalization of the Fourier transform, has different kinds of mathematical definitions [6-9]. This fact enables us to represent signals in different ways. Shih [10] proposed a method to fractionalize the Fourier transform as a composition of the given signal, its ordinary Fourier transform and their reflected versions, only according to three postulates that the FRFT should obey. We generalize the weighted coefficients of the FRFT proposed by Shih to contain two vector parameters. Therefore a generalized FRFT is defined by replacing the weighted coefficients with the generalized ones, which is regarded as the multiple-parameter fractional Fourier transforms (MPFRFT).

Sampling theorem plays a crucial role in signal processing and communications [11-13]. In the sampling problem, the objective is to reconstruct a signal from its samples. For a bandlimited signal, Shannon sampling theorem provides a full reconstruction by its uniform samples with a sampling rate higher than its Nyquist frequency [11]. For non-bandlimited signals, several sampling criteria have been proposed associated with wavelet transform and Wigner distribution function etc. [14-19]. Herein we propose another transform for investigating sampling: the MPFRFT. We show that the MPFRFT may provide additional insights that are not observed with traditional Fourier transform, wavelet transform etc.

The main result from our MPFRFT-based sampling analysis is a generalization of Shannon sampling theorem and Fourier series expansion. The proposed sampling theorems enable us to sample and reconstruct certain signals that are not bandlimited in the conventional Fourier transform domain. In section II, the definitions of one dimensional (1D) and two dimensional (2D) MPFRFT are defined. The sampling analysis based on MPFRFT is given in Section III. An orthogonal basis for the class of bandlimited signals in MPFRFT domain is also given in section III. In section IV, experimental results are proposed to demonstrate the effectiveness of the proposed sampling theorems. Section V concludes this paper.

II. MULTIPLE-PARAMETER FRACTIONAL FOURIER TRANSFORM

Let Θ be an operator. $\Theta : \Theta\big[g(x)\big] = G(u)$. It is generally agreed that the fractional operation Θ^{α} of operation Θ should satisfy the following postulates:

i. Continuity postulate: Θ^{α} should be continuous for all real values α.

ii. Boundary postulate:
$$\Theta^{0}\big[g(x)\big] = g(x), \Theta^{1}\big[g(x)\big] = G(u) \qquad (1)$$

iii. Additivity postulate:
$$\Theta^{\beta}\big\{\Theta^{\alpha}\big[g(x)\big]\big\} = \Theta^{\alpha}\big\{\Theta^{\beta}\big[g(x)\big]\big\} = \Theta^{\alpha+\beta}\big[g(x)\big] \quad (2)$$

According to above postulates, one can fractionalize any operation in different ways.

A. One dimensional MPFRFT

It is well known that the Fourier transform F is periodic with periodicity 4. Therefore, it is reasonable to

assume that any fractional operator F^α of the Fourier transform F is a weighted combination of the four basic operators F^0, F^1, F^2 and F^3. Analogous to Shih's technique [10], we can define the fractional Fourier transform F^α with order α as

$$F^\alpha(x) = \sum_{k=0}^{3} p_k(\alpha) F^k(x) \qquad (3)$$

where the weighted coefficients are the functions of transform order α. According to above three postulates, the coefficients should satisfy the following conditions

i. The coefficients are continuous functions of transform order α;

ii. When α is an integer, the coefficients should be certain values which serve as boundary conditions, see Table I;

iii. The coefficients should satisfy the following coupling equations

$p_0(\alpha+\beta) = p_0(\alpha)p_0(\beta) + p_1(\alpha)p_3(\beta) + p_2(\alpha)p_2(\beta) + p_3(\alpha)p_1(\beta)$

$p_1(\alpha+\beta) = p_0(\alpha)p_1(\beta) + p_1(\alpha)p_0(\beta) + p_2(\alpha)p_3(\beta) + p_3(\alpha)p_2(\beta)$

$p_2(\alpha+\beta) = p_0(\alpha)p_2(\beta) + p_1(\alpha)p_1(\beta) + p_2(\alpha)p_0(\beta) + p_3(\alpha)p_3(\beta)$

$p_3(\alpha+\beta) = p_0(\alpha)p_3(\beta) + p_1(\alpha)p_2(\beta) + p_2(\alpha)p_1(\beta) + p_3(\alpha)p_0(\beta)$ (4)

In order to calculate the values of the coefficients, we take the following transformation

$$\begin{pmatrix} q_0(\alpha) \\ q_1(\alpha) \\ q_2(\alpha) \\ q_3(\alpha) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \begin{pmatrix} p_0(\alpha) \\ p_1(\alpha) \\ p_2(\alpha) \\ p_3(\alpha) \end{pmatrix} \qquad (5)$$

Using this transformation, we get the following four equations with much simpler forms

$$\begin{pmatrix} q_0(\alpha+\beta) \\ q_1(\alpha+\beta) \\ q_2(\alpha+\beta) \\ q_3(\alpha+\beta) \end{pmatrix} = \begin{pmatrix} q_0(\alpha)q_0(\beta) \\ q_1(\alpha)q_1(\beta) \\ q_2(\alpha)q_2(\beta) \\ q_3(\alpha)q_3(\beta) \end{pmatrix} \qquad (6)$$

The solutions of (6) are simple exponential functions. Considering the new boundary conditions for $q_k(\alpha)$ $k = 0, 1, 2, 3$ shown in Table I, we can get the solution of (6) as

$$q_k(\alpha) = \exp\{-2\pi i\alpha(4m_k+1)(4n_k+k)/4\} \qquad (7)$$

where parameter vectors

$$m = (m_0, m_1, m_2, m_3) \in Z^4, n = (n_0, n_1, n_2, n_3) \in Z^4$$

are two arbitrary 4-dimensional integer vectors. Taking the inverse transform of (5), we can obtain the weighted coefficients $p_k(\alpha), k = 0,1,2,3$ as

$$p_k(\alpha) = \frac{1}{4}\sum_{l=0}^{3} \exp\{-2\pi i/4[(4m_l+1)\alpha(4n_l+l)-kl]\} \quad (8)$$

Thus the fractional Fourier transform with order α of signal f, or briefly, the (α)-FRFT of f, can be defined as

$$F^\alpha(x) = \sum_{k=0}^{3} p_k(\alpha, m, n) F^k(x) \qquad (9)$$

with the weighted coefficients $p_k(\alpha, m, n)$ given by (8). Due to the additional freedom degrees provided by parameter vectors m, n, we call this kind of FRFT multiple-parameter fractional Fourier transform (MPFRFT).

Note that when $m = n = (0,0,0,0)$, the MPFRFT reduces to the FRFT proposed by Shih. As shown in Fig. 1, the randomicity of parameter vectors m and n provides us more choices to represent signals.

B. two dimensional MPFRFT

Now we extend the 1D MPFRFT to 2D case. Similar to the 1D case, the 2D (α)-MPFRFT of signal $f(x, y)$ can be defined as

$$F^\alpha[f(x, y)] = \sum_{k=0}^{3} p_k(\alpha, m, n) F^k[f(x, y)] \qquad (10)$$

where F^k denotes the k-order 2D Fourier transform, the weighted coefficients $p_k(\alpha, m, n)$, $k = 0,1,2,3$ are the same as (8).

III. SAMPLING THEOREM ASSOCIATED WITH MPFRFT

A. Sampling theorem associated with one dimensional MPFRFT

A signal f is said to be σ bandlimited in (α)-MPFRFT domain, if there exists a positive σ such that

$$F^\alpha[f](u) = 0, |u| > \sigma.$$

Theorem 1: Suppose $f(t)$ is σ bandlimited in (α)-MPFRFT domain, then $f(t)$ can be uniquely determined by the samples of its ($\alpha-1$)-MPFRFT, and can be completely reconstructed by the following sampling formula:

$$f(t) = \sum_{n} F^{\alpha-1}[f](t_n)\phi(t, t_n) \qquad (11)$$

where $t_n \le n/(2\sigma)$, $\phi(t, t_n)$ is the ($1-\alpha$)-MPFRFT of $\text{sinc}[2\sigma(t-t_n)]$ and $\text{sinc}(t) = \sin(\pi t)/(\pi t)$.

TABLE I.
BOUNDARY VALUES FOR THE COEFFICIENTS $p_k(\alpha)$ AND $q_k(\alpha)$.

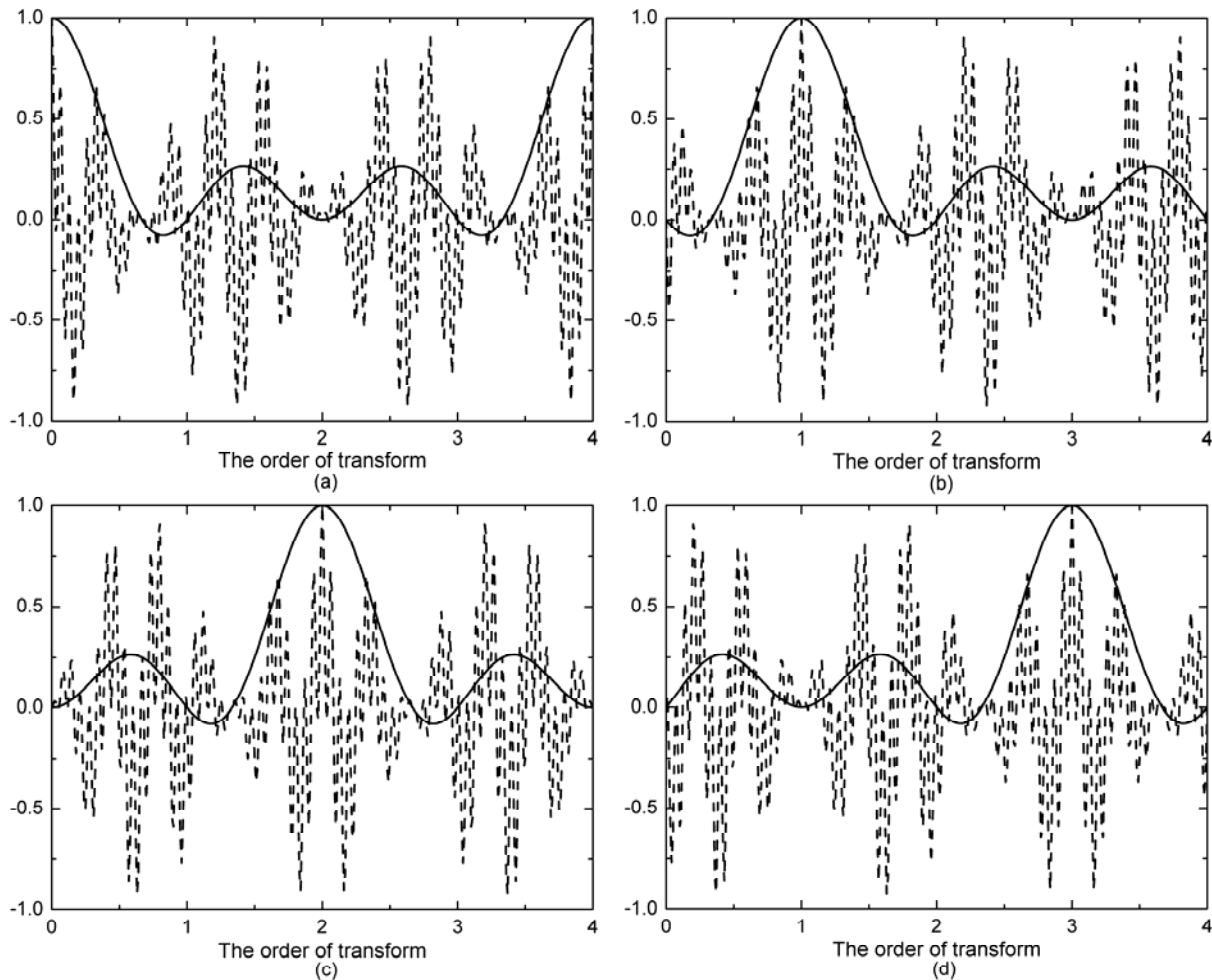| α	$p_0(\alpha)$	$p_1(\alpha)$	$p_2(\alpha)$	$p_3(\alpha)$	$q_0(\alpha)$	$q_1(\alpha)$	$q_2(\alpha)$	$q_3(\alpha)$
0	1	0	0	0	1	1	1	1
1	0	1	0	0	1	$-i$	-1	i
2	0	0	1	0	1	-1	1	-1
3	0	0	0	1	1	i	-1	$-i$

Figure 1. Real parts of the weighted coefficients, plotted as a function of the transform order, with different parameter vectors: Solid curves with parameter vectors $m=n=(0,0,0,0)$ and dashed curves with parameter vectors $m=(1,3,0,2)$ and $n=(3,0,2,1)$. (a) $p_0(\alpha,m,n)$, (b) $p_1(\alpha,m,n)$, (c) $p_2(\alpha,m,n)$, (d) $p_3(\alpha,m,n)$.

Proof: Let $g(u)$ denote the $(\alpha-1)$-MPFRFT of $f(t)$. It is a common engineering practice to model the sampling process by a multiplication with a sampling sequence of

$$\frac{1}{u_s}\text{comb}(\frac{u}{u_s})$$

where u_s is the sampling period. Without loss of generality, we let $u_s > 0$. Here the corresponding sampled signal representation is

$$g_s(u) = g(u)\frac{1}{u_s}\text{comb}(\frac{u}{u_s}) \qquad (12)$$

with

$$\text{comb}(\frac{u}{u_s})=u_s\sum_n \delta(u-u_s)$$

It is known that the Fourier transform of the multiplication of two signals corresponds to a convolution of the Fourier transforms of each signal. Taking into account the properties of the comb function

and the additive property of MPFRFT, we can write the Fourier transform $G_s(w)$ of $g_s(u)$ as

$$G_s(w) = F^\alpha[f](w)*\text{comb}(u_s w)$$

$$= \frac{1}{u_s}\sum_n F^\alpha[f](w-\frac{n}{u_s}) \qquad (13)$$

where superscript $*$ denotes a convolution. Thus the sampling process results in a periodization of the (α)-MPFRFT of f, as illustrated in Fig. 2. Choose ideal low-pass filter

$$R(w)=\begin{cases} 1, & |w| \le \sigma \\ 0, & |w| > \sigma \end{cases} \qquad (14)$$

and $u_s = 1/(2\sigma)$, we have

$$G_s(w)R(w) = 2\sigma F^\alpha[f](w) \qquad (15)$$

By the additive property of MPFRFT, in the $(\alpha-1)$ MPFRFT domain, we have

$$g(u) = g_s(u)*\text{sinc}(2\sigma u)$$

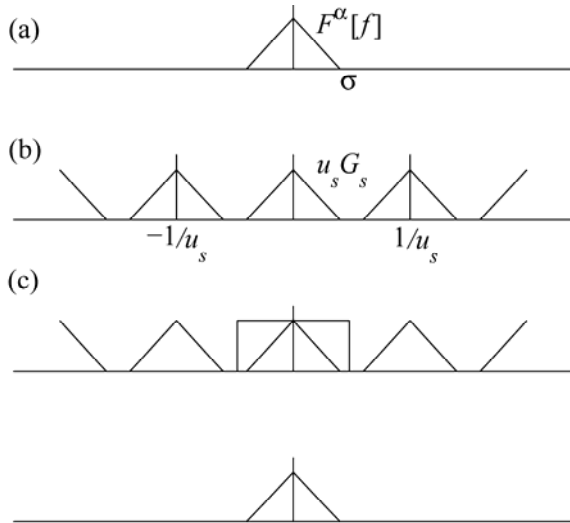$$= \sum_n g(t_n)\text{sinc}[2\sigma(u-t_n)] \qquad (16)$$

Figure 2. Effect of sampling in $(\alpha-1)$ -MPFRFT domain. (a) The (α) -MPFRFT of analog input signal f. (b) Sampling process results in a periodization of the (α) -MPFRFT. (c) The analog signal is reconstructed by ideal low-pass filtering.

Since $f(t)$ is the $(1-\alpha)$ -MPFRFT of $g(u)$, we have

$$
\begin{aligned}
f(t) &= F^{1-\alpha}[g(u)](t) \\
&= \sum_n g(t_n) F^{1-\alpha}\{\text{sinc}[2\sigma(u-t_n)]\}(t) \quad (17)
\end{aligned}
$$

Obviously, when $\alpha=1$ and $m=n=(0,0,0,0)$, Eq. (11) reduces to

$$
f(t) = \sum_n f(t_n)\text{sinc}[2\sigma(t-t_n)] \quad (18)
$$

which means that Theorem 1 includes Shannon sampling theorems as special case. Compared with Shannon sampling theorem, sampling condition is enlarged and the samples are not limited to time domain.

It is interesting to see that when $\alpha=0$ and $m=n=(0,0,0,0)$, Eq. (11) reduces to

$$
f(t) = \begin{cases} (2\sigma)^{-1}\sum_n F(-t_n)e^{-i2\pi t_n t}, & |t| \le \sigma \\ 0, & |t| > \sigma \end{cases} \quad (19)
$$

which means that a timelimited signal can be represented by a Fourier series. Thus, classical Fourier series expansion can also be viewed as a sampling formula in the sense of MPFRFT. Furthermore, the proposed sampling theorem based on MPFRFT gives a continuous conversion from Fourier series expansion to Shannon sampling theorem when $m=n=(0,0,0,0)$ and transform order α ranges from 0 to 1.

Theorem 2: Let H denote the class of signals which are σ bandlimited in (α) -MPFRFT, then we have the following results:

1) The sequence $\{\phi(t,t_n)\}$ forms an orthogonal basis for H ;

2) With respect to above basis, the coordinates of signal are actually the uniform samples of its $(\alpha-1)$ -MPFRFT.

Proof: 1) Let H_1 denote the class of signals which are σ bandlimited in the conventional Fourier transform domain. It has been given before that $\{\phi(t,t_n)\}$ is the $(1-\alpha)$ -MPFRFT of $\text{sinc}[2\sigma(t-t_n)]$. Since the MPFRFT is a unitary mapping of $L^2(R)$ into itself under which H is the image of H_1, it follows that $\text{sinc}[2\sigma(t-t_n)]$ is an orthogonal basis for H_1 if and only if $\{\phi(t,t_n)\}$ is an orthogonal basis for H. It is well known that $\text{sinc}[2\sigma(t-t_n)]$ is an orthogonal basis for H_1. Therefore, $\{\phi(t,t_n)\}$ forms an orthogonal basis for H.

2) From 1) for any $f(t) \in H$, we have

$$
f(t) = \sum_n c_n \phi(t,t_n) \quad (20)
$$

where $c_n, n \in Z$ denote the coordinates of $f(t)$ and can be calculated as

$$
c_n = \left\langle f(t), \frac{\phi(t,t_n)}{\|\phi(t,t_n)\|^2} \right\rangle \quad (21)
$$

where $\langle \cdot,\cdot \rangle$ denotes the inner product and $\|\cdot\|$ is the 2-norm. By the additive property of MPFRFT, it is easy to see that the (α) -MPFRFT $\psi(w)$ of $\{\phi(t,t_n)\}$ is actually the Fourier transform of $\text{sinc}[2\sigma(u-t_n)]$. Thus $\psi(w)$ equals to $(2\sigma)^{-1}e^{-i2\pi t_n t}$ for $|t| \le \sigma$ and equals to zero otherwise. The unitary property of MPFRFT yields that

$$
\|\phi(t,t_n)\|^2 = \|\psi(w)\|^2 = \frac{1}{2\sigma} \quad (22)
$$

So

$$
\begin{aligned}
c_n &= 2\sigma \langle f(t), \phi(t,t_n) \rangle \\
&= 2\sigma \langle F^\alpha[f(t)](w), F^\alpha[\phi(t,t_n)](w) \rangle \\
&= \int_{-\infty}^\infty F^\alpha[f](w) e^{i2\pi t_n w} dw \\
&= F^{\alpha-1}[f](t_n) \quad (23)
\end{aligned}
$$

It can be seen from Theorem 2 that $\{\phi(t,t_n), n \in Z\}$ and their any linear combinations are σ bandlimited in (α) -MPFRFT domain, and can then be completely reconstructed according to sampling formula (11). Obviously, when $\alpha \ne 1$, $\{\phi(t,t_n), n \in Z\}$ are not bandlimited in the conventional Fourier transform domain. Therefore, Theorem 1 provides a full-reconstruction of certain signals that are not bandlimited in the conventional Fourier transform domain.

B. Sampling theorem associated with two dimensional MPFRFT

For notational simplicity, 2D variables (x,y) are denoted as a vector $X=[x\ y]^T$. Let $W(X)$ denote the $(\alpha-1)$ -2D MPFRFT of f. Let us denote the sampled version of $W(X)$ by $W_s(X)$ for which the periodic sampling geometry is indicated by the sampling matrix V as

$$W_s(X) = W(X) \sum_N \delta(X - VN) \qquad (24)$$

where $\delta(X)$ is the 2D impulse function, $N = [n \ m]^T$ and n and m are integers. Sampling matrix

$$V = \begin{bmatrix} x_s & 0 \\ 0 & y_s \end{bmatrix}$$

with x_s and y_s are the distances between samples in the x and y directions, respectively. In the Fourier domain, multiplication corresponds to a convolution. Therefore, by the additive property of MPFRFT, the Fourier transform $\overline{W}_s$ of W_s can be written as

$$\overline{W}_s(U) = F[W_s(X)]$$

$$= F^\alpha[f](U) ** \frac{1}{|\det V|} \sum_N \delta(U - V^{-1}N)$$

$$= \frac{1}{|\det V|} \sum_N F^\alpha[f](U - V^{-1}N) \qquad (25)$$

where $U = [u \ v]^T$, as usual. The double asterisks denotes 2D convolution operator. It can be seen that the Fourier transform of W_s is formed from infinite superposed, shifted replicas of the (α)-MPFRFT of the original signal $f(X)$. The effect of sampling in the $(\alpha - 1)$-MPFRFT domain is illustrated in Fig. 3.

As shown in Fig. 3, the sampling process results in superposed, shifted replicas of the (α)-MPFRFT of the original signal $f(X)$ and the replicas are located at $V^{-1}N$. Therefore, if $f(X)$ is bandlimited in the (α)-MPFRFT domain, say within a band $U \in B = [-b_u, b_u] \times [-b_v, b_v]$, and if the sampling matrix V is chosen to satisfy nonoverlapping replicas in the (α)-MPFRFT domain, say $x_s = (2b_u)^{-1}$ and $y_s = (2b_v)^{-1}$, then $F^\alpha[f](U)$ can be fully recovered by low-pass filtering the Fourier transform $\overline{W}_s$ of W_s. Mathematically, we have

$$\overline{W}_s(U)R(U) = \frac{1}{|\det V|} F^\alpha[f](U) \qquad (26)$$

where

$$R(U) = \begin{cases} 1, & U \in B \\ 0, & \text{else} \end{cases}$$

Therefore, $W(X)$ can be written as

$$W(X) = W_s(X) ** \text{sinc}(V^{-1}X)$$

$$= \sum_N W(VN)\text{sinc}[V^{-1}(X - VN)] \qquad (27)$$

Since f is the $(1 - \alpha)$-MPFRFT of $W(X)$, we thus have

$$f(X) = \sum_N W(VN)\gamma_n(X) \qquad (28)$$

where $\gamma_n(X)$ is the $(1 - \alpha)$-MPFRFT of $\text{sinc}[V^{-1}(X - VN)]$.
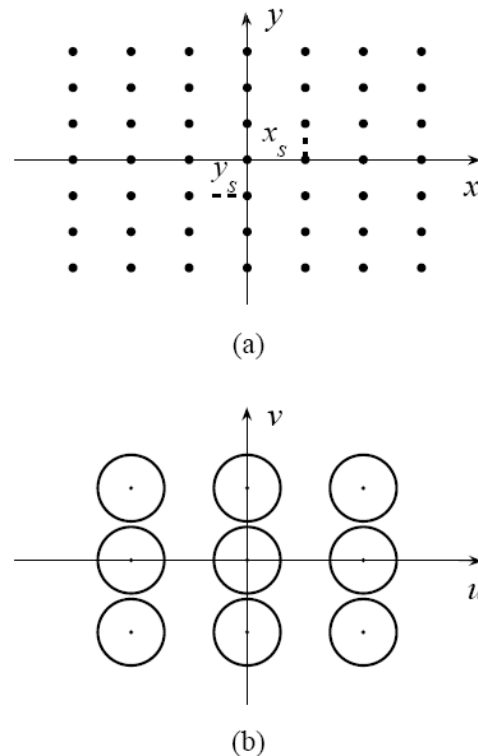

(a)

(b)

Figure 3. Effect of sampling in the $(\alpha - 1) - 2D$ MPFRFT domain. (a) Sampling lattice (spaced by x_s and y_s in the x and y directions, respectively) of the $(\alpha - 1)$-MPFRFT domain, (b) The sampling process results in replications of the (α)-MPFRFT.

The above discussion yields the following sampling theorem.

Theorem 3: Suppose $f(X)$ is $B = [-b_u, b_u] \times [-b_v, b_v]$ bandlimited in (α)-2D MPFRFT domain, then $f(X)$ can be fully reconstructed from its $(\alpha - 1)$-2D MPFRFT domain samples according to formula (28).

IV. SIMULATION EXAMPLES

Figures 4 and 5 depict the results of a numerical experiment demonstrating the effectiveness of sampling formula (11) presented in this work. Figs. 4(a) and (b) show a MPFRFT pair with $\alpha = 0.2$, $m = (1, 3, 0, 2)$ and $n = (3, 0, 2, 1)$. The MPFRFTed signal $F^\alpha[f](w)$ equals to e^{iw}, for $w \in [-1, 1]$, and equals to zero otherwise; that is the original signal $f(t)$ illustrated in Fig. 4(a) is $\sigma = 1$ bandlimited in (0.2)-MPFRFT domain with parameter vectors $m = (1, 3, 0, 2)$ and $n = (3, 0, 2, 1)$. Therefore, from Theorem 1, we can reconstruct $f(t)$ by use of the samples of it (-0.8)-MPFRFT and interpolation functions $\{\phi(t, t_n)\}$. The (-0.8)-MPFRFT domain samples c_n and the reconstructed signal based on Theorem 1 are plotted in Figs. 4(c) and (d), respectively. Several interpolation functions $\{\phi(t, t_n)\}$ in (11) are illustrated in Fig. 5. We can see that an almost perfect reconstruction of $f(t)$ can be obtained.
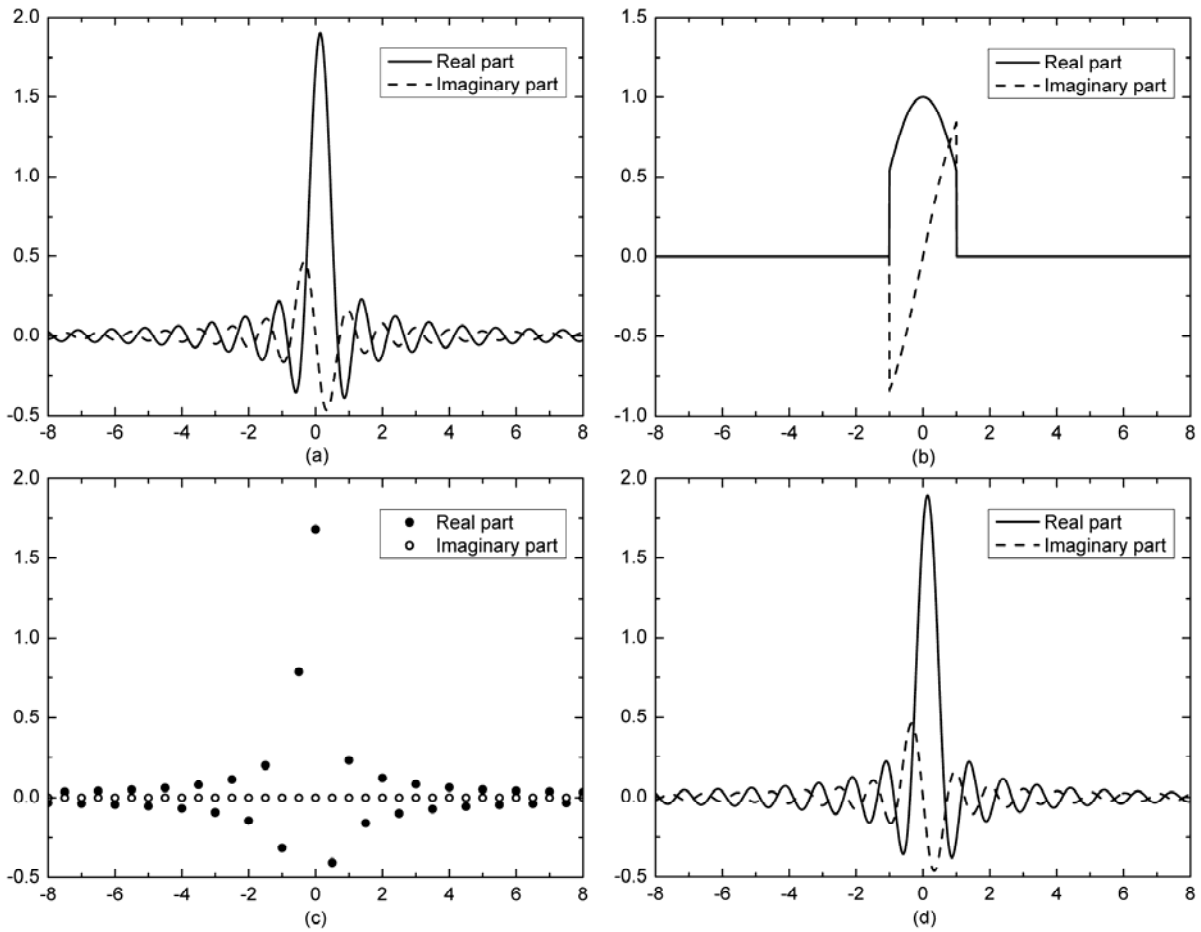
Figure 4. Example of reconstruction of a signal based on Theorem 1. (a) The original signal $f(t)$. (b) The (0.2)-MPFRFT $F^\alpha[f](w)$ of $f(t)$ with $m = (1,3,0,2)$, $n = (3,0,2,1)$. (c) The (-0.8)-MPFRFT domain samples c_n. (d) Reconstructed signal based on Theorem 1. It can be seen that an almost perfect reconstruction is obtained.
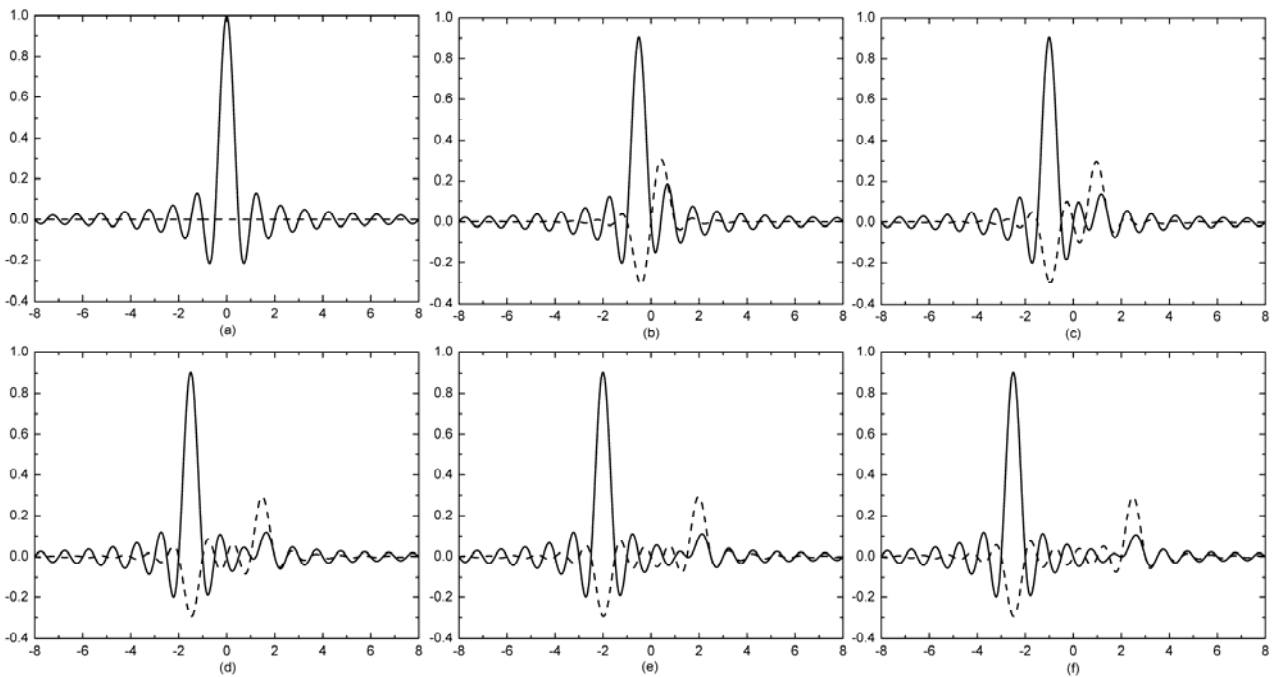

Figure 5. Interpolation functions $\{\phi(t, t_n)\}$ with $\alpha = 0.2$ $\sigma = 1$, and (a) $n = 0$, (b) $n = 1$, (c) $n = 2$, (d) $n = 3$, (e) $n = 4$, (f) $n = 5$. Solid curves stand for real parts and dashed curves stand for imaginary parts.
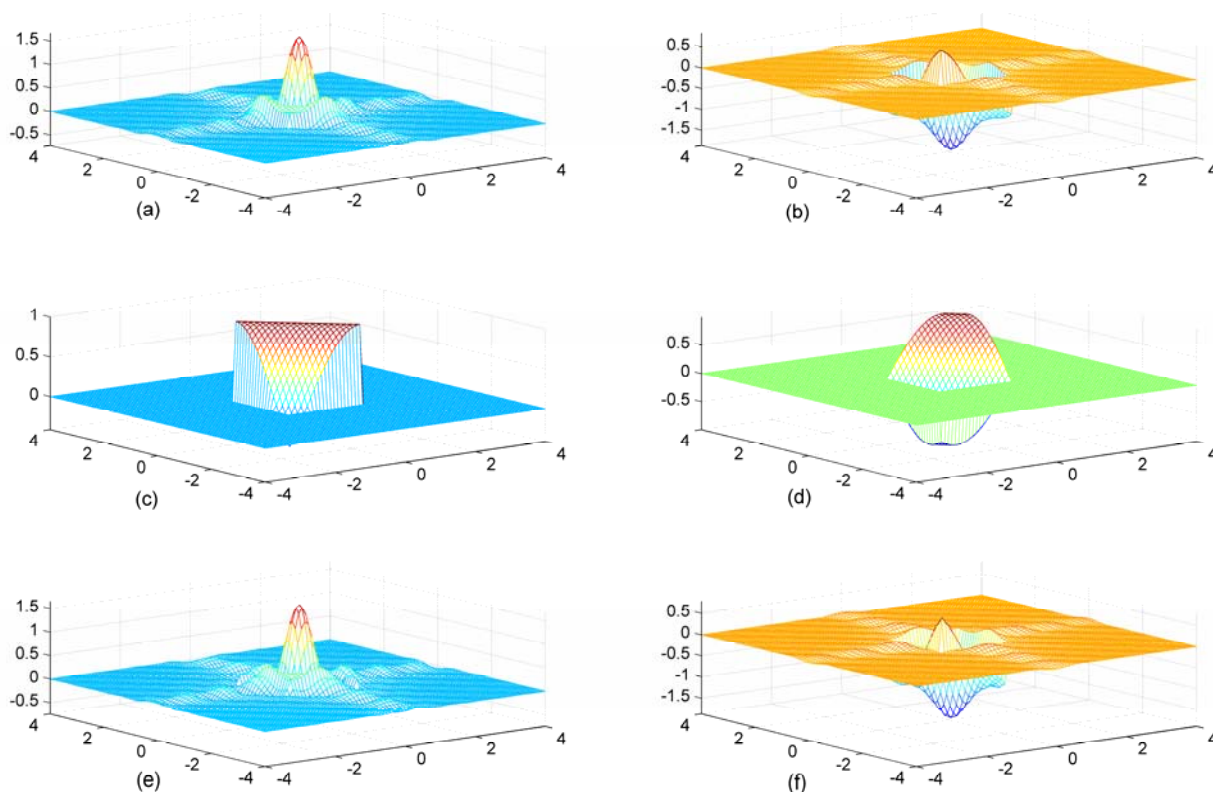
Figure 6. Reconstruction of a 2D signal based on Theorem 3. (a) Real part of the original signal $f(x, y)$, (b) Imaginary part of f, (c) Real part of the (0.6)-MPFRFT $F^{0.6}f$ with parameter vectors $m = (2,5,3,1)$ and $n = (7,4,1,2)$ of f, (d) Imaginary part of $F^{0.6}f$. (e) Real part of the reconstruction signal, and (f) Imaginary part of the reconstruction signal. It can be seen that an almost full reconstruction is obtained.

Simulation example shown in Fig. 6 gives further insight. The real and imaginary parts of a 2D signal are shown in Figs. 6(a) and (b), respectively. The real and imaginary parts of its (0.6)-MPFRFT with parameter vectors $m = (2,5,3,1)$ and $n = (7,4,1,2)$ are shown in Figs. 6(c) and (d), respectively. It can be seen from (c) and (d) that the original signal is $B = [-1,1] \times [-1,1]$ bandlimited in (0.6)-MPFRFT with parameter vectors $m = (2,5,3,1)$ and $n = (7,4,1,2)$. Therefore, from Theorem 3, the original signal can be fully reconstructed by the samples of its (-0.4)-MPFRFT. The (-0.4)-MPFRFT is sampled with a sampling matrix V,

$$V = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

Figs. 6(e) and (f) display the real and imaginary parts, respectively, of the reconstruction signal. It can be seen that an almost perfect reconstruction is obtained.

V. CONCLUSION

In this paper, we first generalized the fractional Fourier transform (FRFT) proposed by Shih to multiple-parameter FRFT (MPFRFT) and extend the 1D MPFRFT to 2D case. Then we proposed a new method for analysis of sampling and reconstruction of signals by use of the MPFRFT. On the basis of observations in MPFRFT domain, we derived a generalization of Shannon sampling theorem and Fourier series extension. The proposed theorem unifies classical Shannon sampling theorem with Fourier series expansion. It also provides a full-reconstruction procedure of certain signals that are not band-limited in the conventional Fourier domain. An orthogonal basis for the class of bandlimited signals in MPFRFT domain is also given, with respect to which the coordinates of signal are actually the samples of its $(\alpha - 1)$-MPFRFT. Experimental results have verified the accuracy and effectiveness of the obtained results.

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Qiwen Ran is a professor at Harbin Institute of Technology. He received the Ph.D degree in computer applications from Harbin Institute of Technology in 1999. His research interests are wavelets theory, fractional Fourier transforms and computer applications.

Hui Zhao is a Ph.D. candidate at Harbin Institute of Technology. Her research interests are fractional Fourier transform, linear canonical transform and signal processing.

Guixia Ge received the Master degree in applied mathematics from Harbin Institute of Technology in 2009. Her research interests are wavelets theory and fractional Fourier transforms.

Jing Ma is a professor at Harbin Institute of Technology. He received the Ph.D degree in physical electronics from Harbin Institute of Technology in 2001. His research interests are satellite optical communication and wavelet optics information processing.

Liying Tan is a professor at Harbin Institute of Technology. She received the Ph.D degree in physical electronics from Harbin Institute of Technology in 2004. Her research interests are optical information processing and satellite optical communication.