Involute of Cubic Pythagorean-Hodograph Curve

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Abstract—Since the mode of Pythagorean-Hodograph curve's derivate is a polynomial, its involute should be a rational polynomial. We study the expression form of PH cubic's involute, discovering that its control points and weights are all constructed by corresponding PH cubic's geometry property. Moreover, we prove that cubic PH curves have neither cusp nor inflection; there are two points coincidence in the six control points of cubic PH curves' involute; the control polygon of cubic PH curve has two edges which are perspectively vertical with two edges in the control polygon of corresponding involute, and we compute the length of the two edge in the involute control polygon; So if we give the cubic PH curve, then we can decide five control points of the six, and we give the vector about the remaining point and another point. When the cubic PH curve is a symmetry curve, its two involute are symmetry, other cases are not.

Index Terms—PH curve, involute, control point, control polygon, turning angle

I. INTRODUCTION

Pythagorean-Hodograph Curve (PH) is brought up by American scientist R.T.Farouki [1] in 1990, whose components of the first derivative vector satisfy the Pythagorean condition. For this particular algebra structure, PH curves are advantages to general curves. For example, arc length is polynomial function 、 offset curve is rational parameter curve, etc [1-4]. PH curves can exactly realize tool compensation in numerical control machining、 numerical control interpolation, etc [5-9]. For this function, PH curves actually obtain widely applications in CAD 、 computer graphics 、 computer vision、 robot technology and motion control, etc.

On the base of the necessary condition for a plane polynomial curve to a PH curve and the geometric structure [1], which is firstly presented by Farouki and other scientists, curves relating to PH curves are studied by many scientist. For example, PH-C curve [10] 、 on control polygons of quartic PH curves [11] 、 Hermite interpolation by PH quintic [12] 、 arc length preserving approximation of circular arcs by quintic PH curves [13], etc. But there is rarely research about the involute of PH curve. Involute is widely applied in cylindrical spur gear、 involute tooth profile, etc [14].

In accordance with the definition of involute and the PH curves' characteristics, we know the involute of PH curve is rational polynomial. After research we know that the involute of cubic PH curve is quintic rational

polynomial; the involute's weight and control points are constituted by cubic PH curve's control polygon's vertexes、 length of the side and the turning angle of adjacent sides; the first two points coincide of the six control points; two sides of the involute's control polygon are respectively perpendicular to two sides of the cubic PH curve's control polygon and can be easily compute the length. So we can easily determine five control points of the six after the cubic PH curve given, regretfully,there is no geometric method to easily determine the remain one control point.

When the cubic PH curve is a symmetrical curve, it's involute is a axial symmetry curve, otherwise, there is no this property.

II. ELEMENTARY KNOWLEDGE

Definition1: If a polynomial parametric curve $p(t) = (x(t), y(t))$ satisfies the Pythagorean condition $x'(t)^2 + y'(t)^2 = \sigma^2(t)$ for some polynomial $\sigma(t)$, then this curve is a PH curve.

Theorem1 [15] :A planar cubic Bézier curve $p(t) = \sum_{i=0}^{3} B_i^3(t) p_i$ is a PH curve if and only if

$$L_2 = \sqrt{L_1 L_3} \qquad (1)$$

$$\theta_1 = \theta_2 \qquad (2)$$

Where L_j is the side length of the control polygon, θ_1 is the angle between the vector $p_0 p_1$ and $p_1 p_2$, θ_2 is the angle between the vector $p_1 p_2$ and $p_2 p_3$.
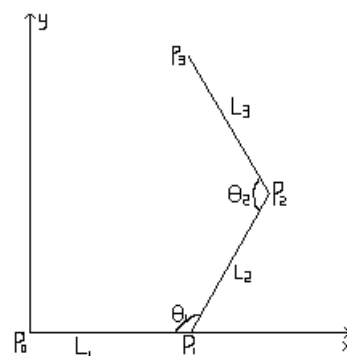

Figure 1   Axis and Geometric parameters

III. INVOLUTE OF CUBIC PH CURVE

Theorem 2:For any regular PH curve $p(t) = (x(t), y(t))$, if $(x'(t), y'(t)) \neq 0$, then this PH curve has none sharp point.

Theorem 3:Cubic PH curve has none sharp point or inflexion

Proof: The control points of Cubic PH curve is $p_i, i = 0 \cdots 3$ $p(t)$, and point M is the intersection of $p_0 p_1$ and $p_2 p_3$.

When the point M is the infinite point on the plane $(\theta = \dfrac{\pi}{2})$ as shown in Fig.2, the control polygon is a convex polygon, by convex persevering we know that cubic PH curve have none inflexion. At the same from the theorem 2 we know it only need us to prove this PH curve is a regular PH curve.

From

$$p'(t) = 3 \sum_{i=0}^{2} B_i^2(t)(p_{i+1} - p_i) .$$

$$p_1 - p_0 = L_1(1,0) .$$

$$p_1 - p_0 = L_1(1,0) .$$

$$p_2 - p_1 = L_2(0,1) .$$

$$p_3 - p_2 = L_3(-1,0) .$$

We get

$$x'(t) = 3[L_1(1-t)^2 - L_3 t^2] .$$

$$y'(t) = 6L_2 t(1-t) .$$

Because $x'(t)$ and $y'(t)$'s highest degree is 2, the highest degree of $\gcd(x'(t), y'(t))$ is 2.

By the definition of regular PH curve, we know its only need to prove $\gcd(x'(t), y'(t)) = 1$ or have the form $(t - a)^2$.

Because $\gcd(x'(t), y'(t))$ has only four form 1、 $(t-a)^2$ 、 $t+b$ and $(t-a)^2 + b, b \neq 0$, in the following we prove the form $t+b$ and $(t-a)^2 + b, b \neq 0$ failed to establish.

If $\gcd(x'(t), y'(t))$ has the form t+b, we know $\gcd(x'(t), y'(t)) = t$ or $\gcd(x'(t), y'(t)) = t - 1$ from the expression of $y'(t)$. If $\gcd(x'(t), y'(t)) = t$, then $L_1 = 0$ from the expression of $x'(t)$,it's a contradiction.

If $\gcd(x'(t), y'(t))$ has the form $(t - a)^2 + b, b \neq 0$, then $\gcd(x'(t), y'(t)) = (t - \dfrac{1}{2})^2 - \dfrac{1}{4}$ from the expression $y'(t)$ .At this time, if $L_1 = L_3$, then $x'(t)$ become to first-degree polynomial, it's a contradiction; if $L_1 \neq L_3$,then the expression of $x'(t)$ become to

$$x'(t) = 3(L_1 - L_3)[(t - \frac{L_1}{L_1 - L_3})^2 + \frac{L_1}{L_1 - L_3} - (\frac{L_1}{L_1 - L_3})^2] . \quad (3)$$

From the (3) we know $\dfrac{L_1}{L_1 - L_3} = \dfrac{1}{4}$ and $\dfrac{L_1}{L_1 - L_3} - (\dfrac{L_1}{L_1 - L_3})^2 = \dfrac{1}{4}$, it's a contradiction.

So cubic PH curve is a regular PH curve

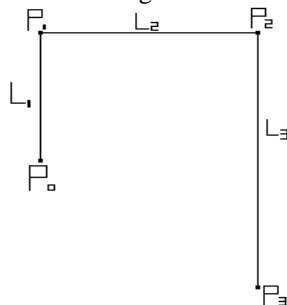Figure 2 Cubic PH curve's control polygon and the intersection of the two sides

When M is a finite point, we let $3\overrightarrow{P_0 P_1} = \lambda \overrightarrow{P_0 M}$, $3\overrightarrow{P_2 P_3} = \mu \overrightarrow{M P_3}$. There are four types of this PH curve about the range of angle θ and the position of point M.

(1):When $0 < \theta < \dfrac{\pi}{2}$, point M is on the extension line of $p_1 p_0$ and $p_2 p_3$ as shown in Fig.3. There is $\lambda < 0, \mu < 0$, from the shape classification and controlled of the plane cubic Bézier curve [16] we know that this cubic PH curve have none sharp point or inflexion.
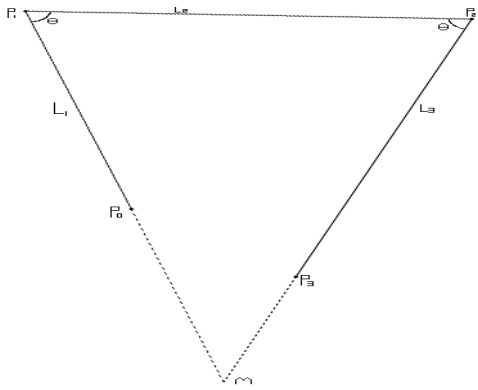
Figure 3   One type of the cubic PH curve's control polygon and the condition about the intersection

Figure 4   One type of the cubic PH curve's control polygon and the condition about the intersection

(2): $0 < \theta < \dfrac{\pi}{2}$, point M is on the extension line of $p_1 p_0$ and between the p_2 and p_3 (when the point M is on the extension line of $p_2 p_3$ and between the point p_0 and p_1, the situation is symmetrical) as shown in Fig.4 There is $\lambda < 0$, $\mu > 0$, so there maybe exit sharp point or inflexion. Form the theorem[16] we know, the conditions of exiting inflexion are: $\lambda < 0, 0 < \mu < 3$; or:

$$\lambda < 0, \mu > 3, (\lambda - 4)(\mu - 4) > 0. \quad (4)$$

The conditions of exiting sharp point are $\lambda < 0, \mu > 0, (\lambda - 4)(\mu - 4) - 4 = 0$ .From the definition of λ and μ we know $\mu > 3$, so only the (4) establish if there exit inflexion.

$$\lambda = \frac{-3L_1}{\dfrac{L_2}{2\cos\theta} - L_1} = \frac{6L_1 \cos\theta}{2L_1 \cos\theta - L_2}.$$

$$\mu = \frac{3L_3}{L_3 - \dfrac{L_2}{2\cos\theta}} = \frac{6L_3 \cos\theta}{2L_3 \cos\theta - L_2}.$$

$$2L_1 \cos\theta - L_2 < 0.$$

$$2L_3 \cos\theta - L_2 > 0.$$

Let λ and μ substitute the expression of $F(\mu, \lambda) = (\lambda - 4)(\mu - 4) - 4$, we get $F(\lambda, \mu) < 0$ there is none sharp point or inflexion.
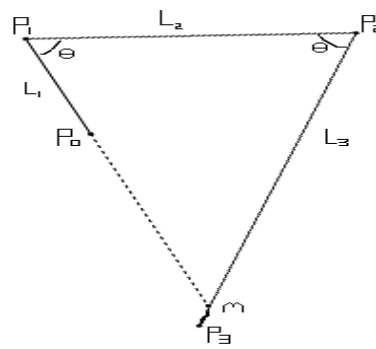
(3): $0 < \theta < \dfrac{\pi}{2}$, point M is between the point p_0 and p_1, as the same time point M is also between the points p_2 and p_3 as shown in Fig.5
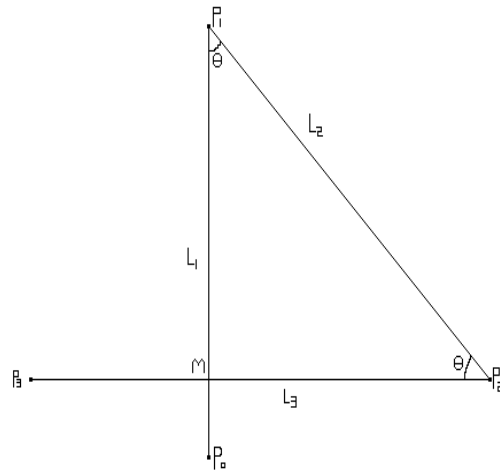

Figure 5   One type of the cubic PH curve's control polygon and the condition about the intersection

Then $\lambda > 3, \mu > 3$ and

$$\lambda = \frac{3L_1}{L_1 - \dfrac{L_2}{2\cos\theta}} = \frac{6L_1 \cos\theta}{2L_1 \cos\theta - L_2}.$$

$$\mu = \frac{3L_3}{L_3 - \dfrac{L_2}{2\cos\theta}} = \frac{6L_3 \cos\theta}{2L_3 \cos\theta - L_2}.$$

$$2L_1 \cos\theta - L_2 > 0.$$

$$2L_3 \cos\theta - L_2 > 0.$$

Let λ and μ substitute the expression of $F(\mu, \lambda) = (\lambda - 4)(\mu - 4) - 4$ we get $F(\lambda, \mu) > 0$, there is none sharp point.

(4) $\dfrac{\pi}{2} < \theta < \pi$, point M is on the extension line

of $p_0 p_1$ and $p_3 p_2$ as Fig. 6, then

$0 < \lambda = \dfrac{3\overrightarrow{p_0 p_1}}{\overrightarrow{p_0 M}} < 3$, $0 < \mu = \dfrac{3\overrightarrow{p_2 p_3}}{\overrightarrow{M p_3}} < 3$. From the

shape classification and controlled of the plane cubic Bézier curve [16] we know the cubic PH curve has none sharp point or inflexion.
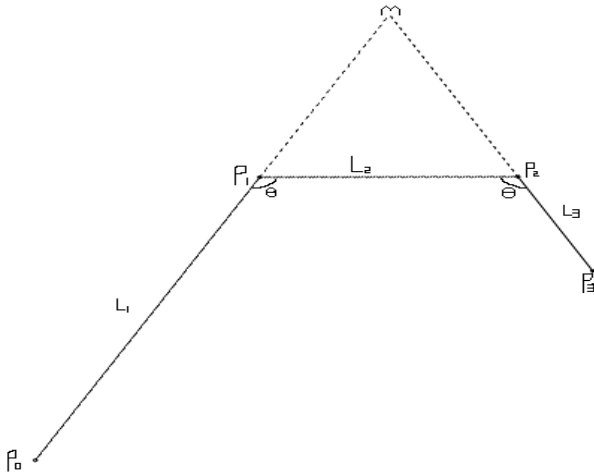

Figure 6   One type of the cubic PH curve's control polygon and the condition about the intersection

A. Definition and Theorem

Definition 2:Curve $Q(t)$ is the involute of curve $p(t)$ of any segment $t \in [0,1]$, then $Q(t)$ satisfies the following conditions: For any point Q of the curve $Q(t)$, there exits a point P of curve $p(t)$, that the line PQ is the tangent line of $p(t)$ where P is the tangent point. The distance P and Q is equal to the arc length form $p(0)$ to $p(t)$ along the curve $p(t)$. From the conditions we can get the expression of $Q(t)$:

$$Q(t) = p(t) - \frac{p'(t)\int_0^t \| p'(s) \| \, ds}{\| p'(t) \|}. \quad (5)$$

Theorem 4: Cubic PH curve's Involute Function:

$$Q(t) = \frac{\displaystyle\sum_{i=0}^{5} B_i^5(t)\sigma_i Q_i}{\displaystyle\sum_{i=0}^{5} B_i^5(t)\sigma_i}.$$

Where $L_i = \| p_{i-1} p_i \|, i = 1,2,3; \theta$ is both the angle between the vector $p_0 p_1$ and $p_1 p_2$ and the angle between the vector $p_1 p_2$ and $p_2 p_3$; the expressions of σ_i and Q_i is as the following:

$$\sigma_0 = 3L_1.$$

$$\sigma_1 = \frac{9L_1 - 6L_2 \cos \theta}{5}.$$

$$\sigma_2 = \frac{9L_1 - 18L_2 \cos \theta + 3L_3}{10}.$$

$$\sigma_3 = \frac{3L_1 - 18L_2 \cos \theta + 9L_3}{10}.$$

$$\sigma_4 = \frac{9L_3 - 6L_2 \cos \theta}{5}.$$

$$\sigma_5 = 3L_3.$$

$$Q_0 = Q_1 = p_0.$$

$$Q_2 = \frac{p_0(3L_1 - 3L_2 \cos \theta + L_3) + p_1(3L_1 - 3L_2 \cos \theta) + p_2(-3L_1)}{3L_1 - 6L_2 \cos \theta + L_3}$$

$$Q_3 = \frac{p_0(L_1 - L_2 \cos \theta + L_3) + p_1(5L_1 - 5L_2 \cos \theta + 2L_3) + p_2(-3L_1) + p_3(-2L_1)}{L_1 - 6L_2 \cos \theta + 3L_3}.$$

$$Q_4 = \frac{p_1(2L_1 - 2L_2 \cos \theta + 2L_3) + p_2(L_1 - L_2 \cos \theta + L_3) + p_3(-3L_1 + L_2 \cos \theta)}{-2L_2 \cos \theta + 3L_3}.$$

$$Q_5 = \frac{p_2(L_1 - L_2 \cos \theta + L_3) + p_3(-L_1 + L_2 \cos \theta)}{L_3}.$$

Proof: If $p(t)$ is expressed by the form $p(t) = (x(t), y(t))$, $p_i = (x_i, y_i)$ in the particular axis as shown in Fig.1, where p_0 is the original point and p_1 at the orthoaxis. According the Theorem 1 we can get

$$L_2 = \sqrt{L_1 L_3} \qquad 、 \qquad p_1 - p_0 = L_1(1,0) \qquad 、 \qquad \text{and } p_3 - p_2 = L_3(\cos 2\theta, -\sin 2\theta).$$

$$p_2 - p_1 = L_2(-\cos\theta, \sin\theta)$$

Immediately we can get the following relationship:

$$x'(t)^2 + y'(t)^2 = 9(L_1 B_0^2(t) - L_2 \cos\theta B_1^2(t) + L_3 B_2^2(t))^2. \qquad (6)$$

Because there is only cubic PH curve's geometric relation in (6), this formula is correct in any axis.

Let

$$q(t) = \sqrt{x'(t)^2 + y'(t)^2} = 3(L_1 B_0^2(t) - L_2 \cos\theta \cdot B_1^2(t) + L_3 B_2^2(t)). \qquad (7)$$

For any segment $t \in [0,1]$, the arc length of curve $p(t)$ is $s(t)$, so we can get the following expression of $s(t)$

$$. s(t) = \int_0^t \sqrt{x'(s)^2 + y'(s)^2}\, ds = \int_0^t q(s)\, ds. \qquad (8)$$

At the moment we can get another expression of (5) in the following:

$$Q(t) = p(t) - \frac{p'(t)s(t)}{q(t)}. \qquad (9)$$

By the calculation of (8), we get

$$s(t) = L_1 B_1^3(t) + (L_1 - L_2 \cos\theta)B_2^3(t) + (L_1 - L_2 \cos\theta + L_3)B_3^3(t).$$

Let $s_0 = 0$ 、 $s_1 = L_1$ 、 $s_2 = (L_1 - L_2 \cos\theta)$ and $s_3 = (L_1 - L_2 \cos\theta + L_3)$, then we get

$$s(t) = \sum_{i=0}^3 B_i^3(t)s_i. \qquad (10)$$

Let $q_0 = 3L_1; q_1 = -3L_2 \cos\theta, q_2 = 3L_3$; then we also get

$$q(t) = \sum_{i=0}^2 B_i^2(t)q_i. \qquad (11)$$

To find a common denominator for (9), we let the numerator is $p(t)q(t) - p'(t)s(t) = \sum_{i=0}^5 B_i^5(t)\widehat{Q}_i$ and the parameters of $\widehat{Q}_i$ have the following expressions:

$$\widehat{Q}_0 = 3p_0 L_1.$$

$$\widehat{Q}_1 = \frac{p_0(9L_1 - 6L_2 \cos\theta)}{5}.$$

$$\widehat{Q}_2 = \frac{p_0(9L_1 - 9L_2 \cos\theta + 3L_3) + p_1(9L_1 - 9L_2 \cos\theta) + p_2(-9L_1)}{10}.$$

$$\widehat{Q}_3 = \frac{p_0(3L_1 - 3L_2 \cos\theta + 3L_3) + p_1(15L_1 - 15L_2 \cos\theta + 6L_3) + p_2(-9L_1) + p_3(-6L_1)}{10}.$$

$$\widehat{Q}_4 = \frac{p_1(6L_1 - 6L_2 \cos\theta + 6L_3) + p_2(3L_1 - 3L_2 \cos\theta + 3L_3) + p_3(-9L_1 + 3L_2 \cos\theta)}{5}.$$

$$\widehat{Q}_5 = p_2(3L_1 - 3L_2 \cos\theta + 3L_3) + p_3(-3L_1 + 3L_2 \cos\theta).$$

$p(t)$ is a cubic curve, so $q(t)$ is a quadratic curve. After (8) reduce to a common denominator, the numerator is quintic polynomial. Because of the difference degree between the denominator and the numerator, we let $q(t)$ become to quintic polynomial $q(t) = \sum_{i=0}^2 B_i^2 q_i = \sum_{i=0}^5 B_i^2 \sigma_i$. On the basis of Degree Raising Properties, we get $(\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) = A_5 A_4 A_3(q_0, q_1, q_2)$

$$A_{n+1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ \dfrac{1}{n+1} & \dfrac{n}{n+1} & 0 & \cdots & 0 & 0 \\ 0 & \dfrac{2}{n+1} & \dfrac{n-1}{n+1} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \dfrac{n}{n+1} & \dfrac{1}{n+1} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

So we get the expression of σ_i in the Theorem 4.

The standard rational form of $Q(t)$ is $Q(t) = \dfrac{\sum\limits_{i=0}^{5} B_i^5(t)\widehat{Q}_i}{\sum\limits_{i=0}^{5} B_i^5(t)\sigma_i} = \dfrac{\sum\limits_{i=0}^{5} B_i^5(t)\sigma_i \dfrac{\widehat{Q}_i}{\sigma_i}}{\sum\limits_{i=0}^{5} B_i^5(t)\sigma_i}$.

Let $Q_i = \dfrac{\widehat{Q}_i}{\sigma_i}$, then the standard rational form of $Q(t)$

becomes to $Q(t) = \dfrac{\sum\limits_{i=0}^{5} B_i^5(t)\sigma_i Q_i}{\sum\limits_{i=0}^{5} B_i^5(t)\sigma_i}$, where the

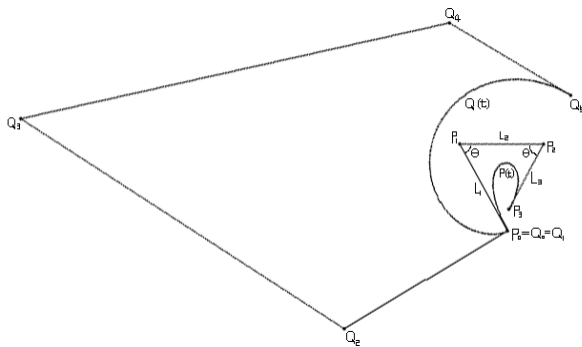expression of Q_i is supposed in the Theorem 4.

This completes the proof.

Figure 7 $0 < \theta < \dfrac{\pi}{2}$,the cubic PH curve $p(t)$ and it's control polygon, the involute of this curve $Q(t)$ and it's control polygon.
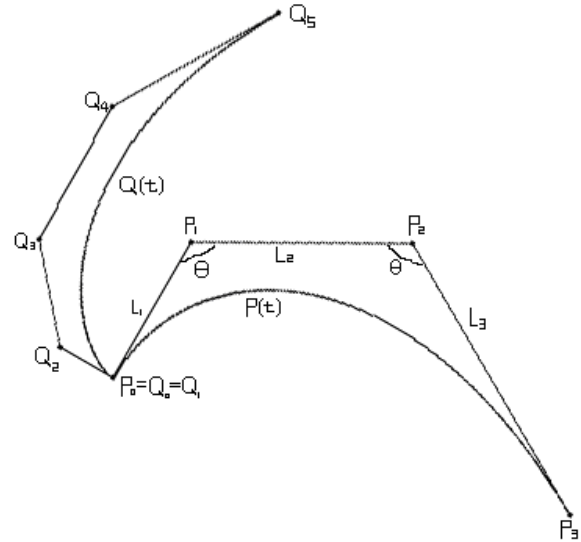

Figure 8 $\dfrac{\pi}{2} < \theta < \pi$,the cubic PH curve $p(t)$ and it's control polygon, the involute of this curve $Q(t)$ and it's control polygon.

B. The Involute's Property

Property 1:There exits an end of Cubic PH curve's involute, whose every direction's derivative is zero(also there exits two control points coincide).

Proof: We compute the derivative of $Q(t)$ about t,we can get

$$Q'(t) = -\frac{p''(t)\int_0^t \| p'(s) \| ds}{\| p'(t) \|} + \frac{p'(t)(p'(t) \bullet p''(t))\int_0^t \| p'(s) \| ds}{\| p'(t) \|^3}. \quad (12)$$

When $t = 0, Q'(0) = 0$ (also $Q(t)'s$ every direction's derivative is zeros).In fact, any curve's involute have this property. Because of the property of Bézier curve that end-side of it's control polygon is tangent to the curve, we know that one side-length of cubic PH curve's control polygon is zero（two control points coincide）. We can find the property from the Theorem 4($Q_0 = Q_1 = P_0$), as shown in Fig.7 and Fig.8.

Property 2:There is a right angle between vector $\overrightarrow{Q_1Q_2}$ and $\overrightarrow{p_1p_2}$,where $\overrightarrow{Q_1Q_2}$ is a control polygon side of this involute and $\overrightarrow{p_1p_2}$ is a control polygon side of Cubic PH curve. we also can get

$$\| Q_2 - Q_1 \| = \frac{3L_1L_2\sin\theta}{\| 3L_1 - 6L_2\cos\theta + L_3 \|}. \quad (13)$$

Proof: Because of

$$Q_2 - Q_1 = Q_2 - P_0 = \frac{-3L_2\cos\theta(p_1 - p_0) - 3L_1(p_2 - p_1)}{3L_1 - 6L_2\cos\theta + L_3}. \quad (14)$$

and $\parallel p_0 p_1 \parallel = L_1$ 、 $\parallel p_1 p_2 \parallel = L_2$,the turning angle between $\overrightarrow{p_0 p_1}$ and $\overrightarrow{p_1 p_2}$ is θ . So we get (10). By the cosine formula we get a right angle between vector $\overrightarrow{Q_1 Q_2}$ and $\overrightarrow{p_1 p_2}$.

This completes the proof.

Property 3:In the control polygon of cubic PH curve's involute, the point Q_5 is on the line $p_2 p_3$ and

$$Q_5 = \frac{p_2(L_1 - L_2 \cos\theta + L_3) + p_3(-L_1 + L_2 \cos\theta)}{L_3} . \quad (15)$$

We know the point Q_5 is on the line $p_2 p_3$ and the angle between vector $\overrightarrow{p_3 Q_5}$ and vector $\overrightarrow{p_2 p_3}$ is a flat angle.

Because of

$$Q_5 - p_3 = \frac{(L_1 - L_2 \cos\theta + L_3)(p_2 - p_3)}{L_3} . (16)$$

we get

$$\parallel Q_5 - p_3 \parallel = L_1 - L_2 \cos\theta + L_3 . \quad (17)$$

This completes the proof.

$$Q_5 - Q_4 = \frac{(2L_1 L_3 - 2L_2 L_3 \cos\theta + 2L_3^2)(p_2 - p_1) + (2L_2 L_3 \cos\theta + 2L_1 L_2 \cos\theta - 2L_2^2 \cos^2\theta)(p_3 - p_2)}{L_3(3L_3 - 2L_2 \cos\theta)} . (19)$$

And the parameters $\parallel p_1 p_2 \parallel = L_2$ 、 $\parallel p_2 p_3 \parallel = L_3$ 、 angle θ between the vector $\overrightarrow{p_1 p_2}$ and $\overrightarrow{p_2 p_3}$, we get

$$\parallel Q_4 Q_5 \parallel = \frac{2L_2 \sin\theta(L_1 - L_2 \cos\theta + L_3)}{\parallel 3L_3 - 2L_2 \cos\theta \parallel} .(20)$$

By the cosineformula we know the angle between the vector $\overrightarrow{Q_4 Q_5}$ and $\overrightarrow{p_2 p_3}$ is a right angle.

This completes the proof.

Property 5:When the cubic PH curve is symmetrical about the perpendicular bisector of the vector $\overrightarrow{p_2 p_3}$,it's involute curve is also symmetrical about the perpendicular bisector of the vector $\overrightarrow{p_2 p_3}$,else there is no this property.

Proof: This result is obvious, as shown in Fig.9 and Fig.10.There ignore the proof.

$\parallel Q_5 - p_3 \parallel = L_1 - L_2 \cos\theta + L_3$,there exits a flat angle between vector $\overrightarrow{p_3 Q_5}$ and vector $\overrightarrow{p_2 p_3}$.

Proof: Because of

Property 4:In the control polygon of cubic PH curve's involute, there exits a right angle between the vector $\overrightarrow{Q_4 Q_5}$ and $\overrightarrow{p_2 p_3}$ and

$$\parallel Q_4 Q_5 \parallel = \frac{2L_2 \sin\theta(L_1 - L_2 \cos\theta + L_3)}{\parallel 3L_3 - 2L_2 \cos\theta \parallel} \quad (18)$$
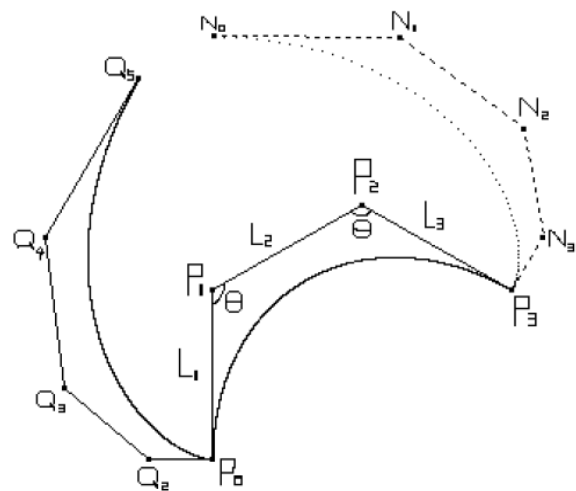
Proof: Because of

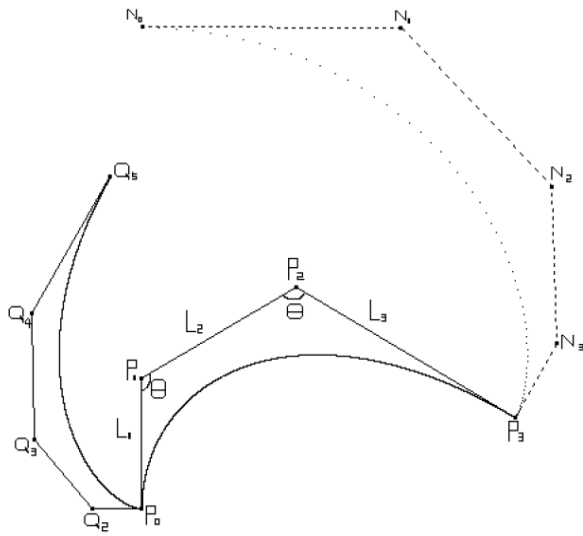Figure 9   When $L_1 = L_3$,The two symmetrical involute of cubic PH curve

Figure 10   When $L_1 \neq L_3$,the two involute of cubic PH curve

IV   CONCLUSION

By this paper we know cubic PH curve's involute can be fully determined by the PH curve's control polygon vertexs、side length and turing angle of adjacent sides. In this paper, some good properties of the control polygon of the cubic PH curve's involute,which include the first two points coincide 、 two sides are respectively perpendicular to two sides of the cubic PH curve's control polygon and can be easily compute the length. The application about the involute mostly is that the circles' involute is applied to machinery, such as gear. Because of the complicated form of general curve's involute,it's hard to apply to the practice. We find that if we fixed two end of a spring, when one end is loosened, the trace of a loosened end is an involute of the spring curve before it is loosened as shown in Fig.4. This paper offer a convenient way of the solution about curve's involute in practice and a basis for further research about involute.
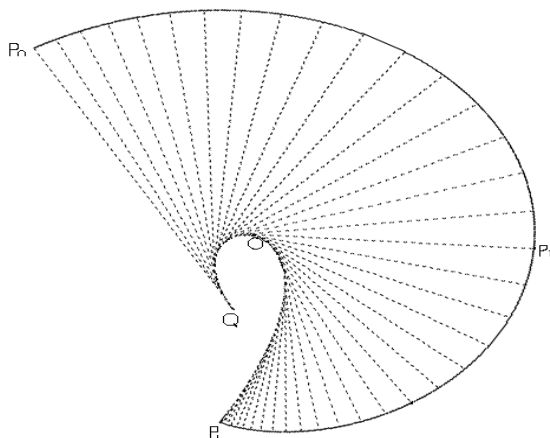

Figure 11   The trace of a loosened end of the arc QOP_0

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